Adiabatic Theory and Wave Action Density

⇒ Quasiparticle Picture

c.f. : Whitham, relevant chapter posted

Wave Adiabatic Theory / Wave Kinetics

- frequently encounter continuum problems with slowly varying parameters $\Rightarrow$ adiabatic theory needed

$\Rightarrow$

- $\boxed{\text{wave kinetic equation}}$ (consequence of Liouville Thm.)

$$\partial_t N + (\underline{v}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k}\cdot\underline{V}) \cdot \partial_{\underline{k}} N$$

$$= C(N)$$; obvious analogy to Boltzmann Eqn.

$N \equiv \mathcal{E}/\omega_{\underline{k}} \equiv$ wave action density (could guess by analogy) / wave quanta density

$\mathcal{E}$: wave energy density

$$\mathcal{E} = \frac{\partial}{\partial\omega}(\omega \epsilon_{\underline{k}})\Big|_{\omega_{\underline{k}}} \frac{|E_{\underline{k}}|^2}{8\pi}$$, for e.s. waves

$N \Leftrightarrow f$

characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial\omega}{\partial k}\hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}}(\omega + \underline{k}\cdot\underline{V})$$

refraction by shear

refraction by parametric variation

- need:

$$\omega < \frac{1}{\lambda}\frac{d\lambda}{dt} \qquad \lambda \equiv \text{parameter}$$

$\rightsquigarrow$ $\boxed{\text{space and time scale separation}}$

$$\frac{1}{N}(\underline{v}_{gr}\cdot\nabla N) \ll \omega \quad \Rightarrow \quad \underline{\nabla}\cdot\underline{v}_{gr} < \omega$$

Transport Eqn - QM

$$\frac{\partial n}{\partial t} + v_{gr} \cdot \nabla n - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} n = C(n)$$

$$\frac{\partial n}{\partial t} + \frac{\partial \hbar\omega}{\partial \hbar \underline{k}} \cdot \nabla n - \frac{\partial \hbar\omega}{\partial x} \cdot \nabla_{\hbar k} n = C(n)$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial \epsilon}{\partial \underline{p}} \cdot \nabla n - \frac{\partial \epsilon}{\partial x} \cdot \nabla_{p} n = C(n)}$$

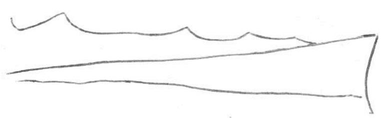
Valid for:

$$\frac{1}{\epsilon} \frac{\partial \epsilon}{\partial x} < 1/\lambda_{DB}$$

$$\lambda_{DB} = \hbar/p.$$

C(N) → interaction with comparable scale.
ignore here.

Examples :



- beach
- linear theory of Langmuir turbulence
  i.e. when will phonon grow ?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasmon energy → net impact ?!
- drift waves and sheared flow.
→ transport equation, super-fluids $\quad N = \frac{\varepsilon}{\omega}$

→ dynamics ?!

Fundamentals of Wave Kinetics

→ where does conservation of action emerge from ?

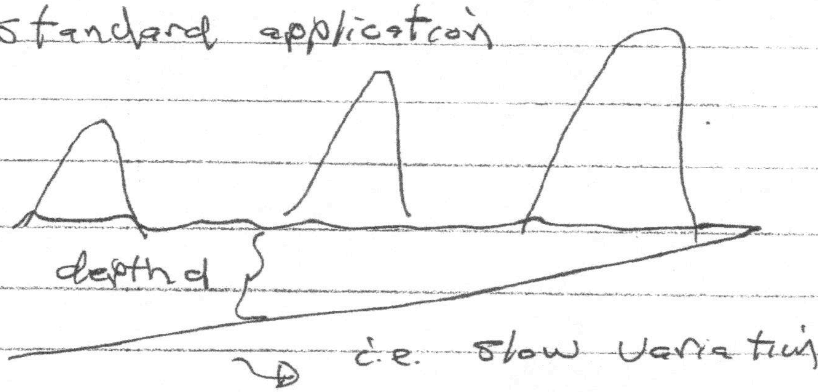
→ answer: $\boxed{\text{phase symmetry, underlies wave kinetics}}$ [of wave train]

→ How show, beyond approach by analogy ?
→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

⇒ standard application

⇒



{ waves in shallow water

$$\frac{1}{d}\frac{d}{dx}d(x) \ll k$$

- in flux of wave energy

- depth $H(x,y)$ decreases

⇒ wave amplification, breaking.

- Derivation

Consider a system, [like ideal fluid MHD, acoustics] which can be described in terms of displacement $\underline{\xi}$;

i.e. $\underline{\xi} = \mathrm{re}\left\{ A e^{i\overset{\text{phase}}{\phi}} + A^* e^{-i\phi} \right\}$

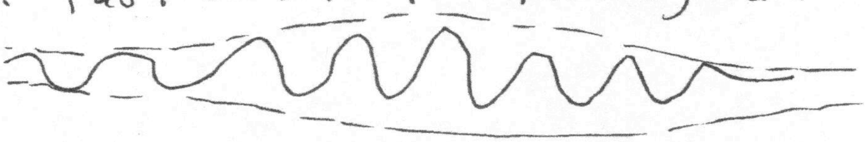
i.e. acoustics!

displacement can be viewed as excitation level

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x}\, \mathcal{L}(\xi)$$

- Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, a la WKB:



$$S = \int dt \int d\underline{x}\, \mathcal{L}(\omega, \underline{k}, \underset{\text{amplitude}}{a})$$

$$\begin{cases} \underline{k} = \nabla\phi \\ \omega = -\partial_t\phi \end{cases}$$

$$= \int dt \int d\underline{x}\, \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

- neglect all corrections to eikonal theory. (no higher order WKB).

- where L corresponds to period-averaged Lagrangian
- ϕ undetermined to const $\rightarrow$ phase symmetry!

$\therefore$ to vary:

$$\left.\begin{aligned} \delta S/\delta a &= 0 \\ \delta S/\delta \phi &= 0 \end{aligned}\right\} \rightarrow \underline{\underline{2\text{ eqns}}}$$

$$\left[\, G(k,\omega) \Leftrightarrow G \,\right]$$

Now, in linear theory:

- $$\mathcal{L} = G(\omega, k)\, a^2 \qquad \left\{ \begin{aligned} & G(\omega, \underline{k}) = 0 \quad \underline{\underline{\text{dispn}}} \\ & \omega^2 = k^2 c_s^2 \end{aligned} \right.$$

continuum, i.e. acoustics

i.e. for ~~[illegible]~~, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho\, \dot{\underline{\xi}}^2 - \frac{1}{2}\rho \left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

$\hookrightarrow$ $\left\{ \begin{array}{l} \text{eikonal form of} \\ \text{stiffness matrix} \\ (\rightarrow \underline{\text{potential}}\ \underline{\text{energy}}) \end{array} \right.$

$$\Rightarrow \underline{\xi} \cdot \underline{\underline{D}} \cdot \underline{\xi}$$

$\downarrow$

$\underline{\underline{D}}(k, \omega, \theta)$, as for linear waves

if: $\underline{\xi} = \underline{A} e^{i\phi} + \underline{A}^{*} e^{-i\phi}$

$$\Rightarrow \quad G(\omega, \underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2\right]$$

Now, 1) $\delta S/\delta a = 0$

$$\Rightarrow \quad G(\omega, k) = 0 \quad \rightarrow \text{dispn relation}$$

but

$$G(\omega, k) = \rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2$$

$$= \rho\omega^2 - \underbrace{D^2}_{\text{stiffness fctn.}}$$

$\Rightarrow$ dispn. relation

2) $\delta S/\delta\phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial\mathcal{L}}{\partial(-\phi_t)}\delta(-\phi_t) + \frac{\partial\mathcal{L}}{\partial(\phi_{\underline{x}})}\delta(\phi_{\underline{x}}) \right\}$$

end pts fixed, i.b.p

$$= \int dx \int d^3x \left\{ \partial_t\left(\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\underline{\nabla}\phi)}\right)\right\}\delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\right) - \underline{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0 \quad \text{conservation eqn.}$$

Now, have: $G(k, \omega) = 0$ (dispn. reln.)

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}}\cdot d\underline{k} = 0$$

$$\therefore \underline{v}_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$ (akin ω)

$$\boxed{\partial_t\left((\partial G/\partial \omega)a^2\right) + \underline{\nabla}\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\,\frac{\partial G}{\partial \omega}a^2\right] = 0}$$

and so $N \equiv \underbrace{\frac{\partial G}{\partial \omega}\;a^2}$

$$\boxed{\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left(\underline{v}_{gr} N\right) = 0}$$ cons. eqn.

(N not yet action)

though:

$$G = \rho\omega^2 - \Omega^2$$

$$\partial G/\partial \omega = 2\rho\omega = 2\rho\omega$$

$$\frac{\partial G}{\partial \omega}a^2 \to \mathcal{E}/\omega\,;$$

i.e. $\omega\frac{\partial G}{\partial \omega}a^2 = 2\rho\frac{\omega^2}{\omega}a^2 = \frac{2(KE)}{\omega} = \mathcal{E}/\omega \to$ action density

Also note energy conserved $\Leftrightarrow$ G invariant to time translations $(\mathcal{L} = G a^2)$.

Thus, Noether's thm $\Rightarrow$ there exists an energy conservation eqn.

Can expect form (see previous)

$$\partial_t \mathcal{E} + \underline{\nabla} \cdot \underline{S} = 0$$

Question: What is N? relation to $\mathcal{E}$, $\underline{S}$?

<u>an argument</u>

Now, have:

$$\mathcal{L} = G(k, \omega) a^2$$

$$\partial \mathcal{L} / \partial a = 0 \Rightarrow G(k, \omega) = 0$$

$$\partial \mathcal{L} / \partial \phi = 0 \Rightarrow \partial_t \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

and also realize:

$$\underline{\nabla} \times \underline{k} = 0 \quad , \quad \underline{k} = \underline{\nabla} \phi$$

$$\frac{\partial \underline{k}}{\partial t} = \frac{-\partial \omega}{\partial \underline{x}} \quad \text{i.e.} \quad \partial_t \underline{\nabla} \phi = -\underline{\nabla} (-\partial_t \phi)$$

$$= -\underline{\nabla} \omega$$

expect: $\dfrac{\partial \mathcal{L}}{\partial \omega} = N$

so $\omega \, \partial \mathcal{L} / \partial \omega = \mathcal{E}$.

Now, assert:

$$\partial_t\left(\omega\frac{\partial \mathcal{L}}{\partial\omega}\right) + \underline{\nabla}\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial \underline{k}}\right) = 0$$

as energy eqn.

To check:

→ add zero (i.e. $\mathcal{L} = \underset{\downarrow \, =0}{\sigma} a^2$)

$$\partial_t\left(\omega\frac{\partial\mathcal{L}}{\partial\omega} - \mathcal{L}\right) + \underline{\nabla}\cdot\left[-\omega\frac{\partial\mathcal{L}}{\partial\underline{k}}\right]=0$$

then

$$(\partial_t\omega)\mathcal{L}_\omega + \omega\,\partial_t\mathcal{L}_\omega - \partial_t\mathcal{L} + \underline{\nabla}\cdot\left(-\omega\frac{\partial\mathcal{L}}{\partial\underline{k}}\right)=0$$

but

$$\partial_t\mathcal{L}_\omega = \underline{\nabla}\cdot(\mathcal{L}_{\underline{k}})$$

phase variation eqn

$$(\mathcal{L}_\omega)(\partial_t\omega) + \cancel{\omega\,\underline{\nabla}\cdot\mathcal{L}_{\underline{k}}} - \omega\left(\cancel{\underline{\nabla}\cdot\mathcal{L}_{\underline{k}}}\right)$$
$$- (\partial\mathcal{L}/\partial\underline{k})\cdot\underline{\nabla}\omega - \partial_t\mathcal{L} = 0$$

$$\partial_t\underline{k} = -\underline{\nabla}\omega$$

$$(\partial_t\omega)(\mathcal{L}_\omega) + (\partial_t\underline{k})\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) - \frac{\partial\mathcal{L}}{\partial t} = 0$$

is an identity.

so $$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right\} + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

but $\mathcal{L} = 0$

so

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \underline{\nabla} \cdot \left(\underline{v_{gr}} \, \omega \frac{\partial \mathcal{L}}{\partial \omega} \right) = 0$$

$\Rightarrow$ energy equation.

$$\mathcal{E} = \omega \frac{\partial \mathcal{L}}{\partial \omega} = \text{wave energy density}$$

so then

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega = N$$

$\rightarrow$ wave action density

$\rightarrow$ by construction, an adiabatic invariant for wave packet.

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Poynting Form

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\partial \omega} \rightarrow \left\{ \begin{matrix} \text{wave} \\ \text{energy density} \end{matrix} \right.$$

so

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega \rightarrow \left\{ \begin{matrix} \text{wave} \\ \text{action density} \end{matrix} \right.$$

$$= N(\underline{k}, \underline{x}, t) \rightarrow \sim \text{adiabatic invariant for wave packet.}$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \nabla N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial \omega / \partial \underline{x} \, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$ (i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{V}_{gr} N \right] = 0$$

Observe:

$\rightarrow$ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{V}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

$\rightarrow$ continuity-type equation in $\underline{x}$-space, for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (\underline{V}_{gr} N) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

$(\omega = D)$

Now $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, following packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr} \cdot \nabla \underline{k}$$

$$= -\frac{\partial \omega}{\partial \underline{x}} + \underbrace{\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}}}_{\text{index}}$$

$\therefore$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial D}{\partial \underline{x}}$$ agree!

$\rightarrow$ Now, can convert from N to E!

i.e. $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{rays} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\bigg|_{\text{rays}} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\bigg|_{\text{rays}} = 0$$

Now $\displaystyle \frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{d\underline{k}}{dt}$

From eikonal eqns:

$$= \partial_t\omega + \cancel{\frac{\partial\omega}{\partial\underline{x}}\cdot\frac{\partial\omega}{\partial\underline{k}}} - \cancel{\frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\omega}{\partial\underline{x}}}$$

so if $\partial_t\omega = 0$ energy conserved

$\therefore$ $\displaystyle \frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0.$

so $\displaystyle \partial_t\mathcal{E} + \underline{V}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_{\underline{k}}\mathcal{E} = 0$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{V}_{gr}\,\mathcal{E}\right] = 0}$$

conserved energy density

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot [\underline{v}_{gr} \mathcal{E}] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

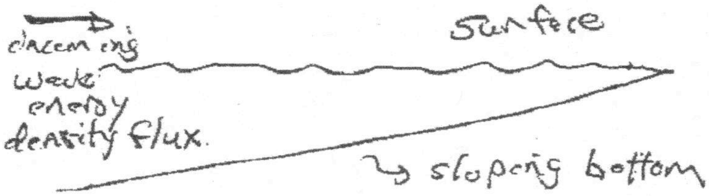
$$\Rightarrow \boxed{\underline{\nabla} \cdot [\underline{v}_{gr} \mathcal{E}] = 0}$$

incompressible wave energy flux!

$\Rightarrow v_{gr}$ drops $\Rightarrow$ $\mathcal{E} \uparrow \Rightarrow$ blocking, breaking

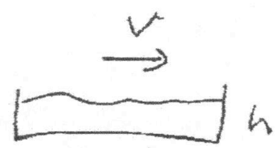
③ The beach....

Consider:



$H = H(x)$

Now, in shallow water $(\lambda > H)$



$$
\begin{cases}
\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(\underline{v} h) = 0 & \text{(continuity)} \\[2ex]
\dfrac{\partial v}{\partial t} + \dfrac{v \cdot \partial v}{\underline{\partial x}} = -g \dfrac{\partial h}{\partial x} & \text{(mom.)}
\end{cases}
$$

~~slope~~ slope

shallow water eqns.

$\dfrac{\partial h}{\partial x} \longrightarrow$ replaces pressure

$v = \cancel{v_0} + \tilde{v}$, $h = H + \tilde{h}$

$$\Rightarrow \quad -i\omega \tilde{h} + ikH\tilde{v} = 0$$
$$-i\omega \tilde{v} = -ikg\tilde{h}$$

$\therefore \Rightarrow \omega^2 = k^2 gH$ is dispersion relation

$gH \leftrightarrow c_s^2$

$\Rightarrow$ analogy with acoustics is obvious.

$h \leftrightarrow \rho$ $\quad c_s^2 = gH$

$v \leftrightarrow v$ $\quad$ etc.

energy $\Rightarrow$

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$$\Rightarrow (1) \times \tilde{v} + (2) \times \left(g \frac{\tilde{h}}{H}\right)$$

$$\therefore \quad \frac{\partial}{\partial t} \tilde{v}^2 = -g \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial}{\partial t} \tilde{h}^2 = -g \frac{\cancel{H}}{\cancel{H}} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

wave energy density flux

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H}\right) + \frac{\partial}{\partial x}\left(g \tilde{h} \tilde{\underline{v}}\right) = 0$$

is energy theorem

$$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H} \text{ is wave energy density.}$$

$\omega/k = (gH)^{1/2}$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot (v_{gr} \mathcal{E}) = 0$$

$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty\,\mathcal{E}_\infty = \underline{I}$

$\downarrow$ incoming wave flux

$V_g = \sqrt{gH(x)}$

$\hookrightarrow$ shallow water waves have zero dispersion

$\therefore \quad \sqrt{g(H(x))}\;\mathcal{E}(x) = \underline{I}$

as $x \to$ shore $V_g \downarrow$ so wave energy must increase.

Now $\mathcal{E}(x) = \dfrac{\overline{\tilde{v}^2}}{2} + \dfrac{g\,\overline{\tilde{h}^2}}{2H} \cong \dfrac{g\,\tilde{h}^2}{\cancel{2}H}$

$$\sqrt{gH(x)}\;\frac{g\,\tilde{h}^2}{\cancel{2}H(x)} = I$$

$$\frac{\tilde{h}^2}{H(x)^2} = \frac{\cancel{2}I}{(\sqrt{g})^3}\left(\sqrt{H(x)}\right)^{-3}$$

i.e.

$$\boxed{(\tilde{h}/H)^2 \sim (\text{const})\,I\,/\,(H(x))^{3/2}}$$

i.e. $\tilde{h}/H \to 1 \Leftrightarrow$ breaking $\Leftarrow$ as $\underline{H(x) \text{ drops}}$.

N.B.:

→ if know bottom profile, can deduce displacement profile, and approximate breaking point.

→ 2 D bottom contours ⇒ wave refraction

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial \underline{x}} = -kg\left(\frac{\partial}{\partial \underline{x}} H(x,y)\right)$$

i.e. wavefronts tend to align with bottom contours approaching shore.